SELECTION OF APPROXIMATE SOLUTIONS OF INVERSE HEAT-CONDUCTION PROBLEMS

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On the basis of the machine experiments, we analyze the various conditions used to select the order of approximation to the solutions of inverse heat-conduction problems by iterative regularization methods.

The contemporary approach to the diagnostics and identification of unsteady heat-exchange processes is based on the analysis of inverse problems. In the overwhelming majority of cases, the inverse heat-exchange problems are incorrectly posed or ill determined and regularization algorithms are necessary [1, 2]. As shown in [2, 3], effective solution algorithms of various types can be obtained with the help of iterative regularization. This method is based on gradient algorithms generating regularizing families of operators with the iteration number as a parameter.

The iterative process of solving incorrectly posed inverse problems develops in two stages, called here the regular and nonregular stages. The regular stage is characterized by a monotonic approach of the approximation to the required solution of the problem. The basic features of the recovered function are successively refined in this stage. In the second stage, better approximations cannot be guaranteed, as undesirable oscillations usually develop gradually due primarily to errors in the initial data.

Therefore, the main question in applying the iterative method is how to determine when to stop, i.e., how one determines the number of iterations which is approximately on the boundary dividing two stages discussed above.

Currently, the most widely used criterion for stopping the iterative process is an error cutoff, where an error functional to be minimized reaches a value corresponding to the total errors in the experimental data [2, 4, 5]. Rigorous results on the stability conditions for the approximations in the gradient method using this approach can be found in [5, 6].

A successful application of the error cutoff criterion requires an a priori knowledge of the error in the initial data. But in practice this requirement is far from always fulfilled. Therefore, the problem of finding other methods of choosing an optimal level of approximation is of interest.

We consider this question using an example of an iverse boundary-value problem involving heat conduction, although we assert that the method described below can also be used for other types of inverse problems.

Let it be required to find functions $q(\tau)$ and $T(x, \tau)$ from the conditions

$$c\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right),\quad x\in(0,\ b),\quad \tau\in(0,\ \tau_m), \tag{1}$$

$$T(x,\ 0)=0,\quad \frac{\partial T(b,\ \tau)}{\partial x}=0,\quad -\lambda\frac{\partial T(0,\ \tau)}{\partial x}=q(\tau), \tag{2}$$

$$T(x_1,\ \tau)=f(\tau),\quad 0\leqslant x_1\leqslant b, \tag{3}$$

where $c = c(T)$, $\lambda = \lambda(T)$, and $f(\tau)$ are known functions.

This statement of the problem corresponds to two possible practical situations, when the actual process of heat transport in the body can be represented as a heating (or cooling) of an infinite plate or rod by a heat flux of unknown density $q(\tau)$, uniformly distributed over the boundary $x = 0$ with the other boundaries of the body being thermally insulated.

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As initial experimental information, the temperature at a certain point in the body $x = x_1$ is given. But the measurements have errors so that the function $f(\tau)$ includes an (unknown) exact part $\bar{f}(\tau)$ and an error $\delta f(\tau)$:

$$f(\tau) = \bar{f}(\tau) + \delta f(\tau).$$

We first consider the solution of the direct heat-conduction problem with boundary conditions of the second kind, so that we assume that the function $q(\tau)$ in (1) and (2) is given.

The temperature field inside the body is computed by numerical methods based on a finite-difference approximation of the quasilinear differential problem (1), (2). Using a uniform grid and the implicit method of [7], we obtain a system on nonlinear algebraic equations to order $O(\Delta\tau + h^2)$, which can be written in the form:

$$\frac{c_1^j \rho}{\Delta\tau}(T_1^j - T_1^{j-1}) = \frac{2\lambda_{1+0,5}^j}{h^2} + \frac{2}{h} q_j,$$

$$\frac{c_i^j \rho}{\Delta\tau}(T_i^j - T_i^{j-1}) = \frac{1}{h^2}[\lambda_{i+0,5}^j (T_{i+1}^j = T_i^j) - \lambda_{i-0,5}^j (T_i^j - T_{i-1}^j)], \tag{4}$$

$$\frac{c_N^j \rho}{\Delta\tau}(T_N^j - T_N^{j-1}) = -\frac{2}{h^2}\lambda_{N-0,5}^j (T_N^j - T_{N-1}^j),$$

where

$$c_i^j = \left(\frac{\overset{s-1}{T_i^j} + \overset{s}{T_i^{j-1}}}{2}\right); \quad \lambda_{i\pm 0,5}^j = \lambda\left(\frac{\overset{s-1}{T_i^j} + \overset{s-1}{T_{i\pm 1}^j}}{2}\right),$$

and s is the number of the current iteration, which is omitted in the basic equations of the difference solution of the direct problem.

For the coefficients given above, the system of equations (4) is linear. The system is well determined and can be solved efficiently by iteration on the thermophysical characteristics.

In the solution of the inverse problem, we use an iterative method of minimizing the functional

$$J = \int_0^{\tau_m} [T(x_1, \tau) - f(\tau)]^2 d\tau, \tag{5}$$

based on the method of conjugate gradients [8]. The unknown function $q(\tau)$ is represented as a piecewise-linear approximation which can be represented as an M-dimensional vector

$$\mathbf{q}(\tau) = \{q(\tau_1), q(\tau_2), \ldots, q(\tau_M)\}.$$

The iteration sequence is set up with the help of the method of conjugate gradients according to the formula

$$\mathbf{q}_{k+1} = \mathbf{q}_k - \beta_k \mathbf{p}_k. \tag{6}$$

The gradient functional $J'(q)$ is necessary to get the direction of the slope $\mathbf{p}_k$ and is calculated using a method based on Lagrange multipliers [9]. This method, as applied to the solution of inverse heat-conduction problems, has been considered in [10, 11]. In the nonlinear case the use of this method leads to a conjugate problem described by the system of algebraic equations [11]:

$$-\frac{\rho}{\Delta\tau}(\bar{c}_1^{j+1} a_1^{j+1} - \overset{+}{c}_1^j a_1^j) = \frac{\overset{+}{\lambda}_1^j}{h^2}(a_2^j - 2a_1^j) + \frac{\partial J}{\partial T_1^j},$$

$$-\frac{\rho}{\Delta\tau}(\bar{c}_2^{j+1} a_2^{j+1} - \overset{+}{c}_2^j a_2^j) = \frac{\overset{+}{\lambda}_2^j}{h^2}(a_3^j - a_2^j) - \frac{\bar{\lambda}_1^j}{h^2}(a_2^j - 2a_1^j) + \frac{\partial J}{\partial T_2^j},$$

$$-\frac{\rho}{\Delta\tau}(\bar{c}_i^{j+1} a_i^{j+1} - \overset{+}{c}_i^j a_i^j) = \frac{\overset{+}{\lambda}_i^j}{h^2}(a_{i+1}^j - a_i^j) - \frac{\bar{\lambda}_{i-1}^j}{h^2}(a_i^j - a_{i-1}^j) + \frac{\partial J}{\partial T_i^j},$$

$$-\frac{\rho}{\Delta\tau}\left(\overline{c}_{N-1}^{j+1}a_{N-1}^{j+1}-\overset{+}{c}_{N-1}^{j}a_{N-1}^{j}\right)=\frac{\overset{+}{\lambda}_{N-1}^{j}}{h^2}\left(2a_N^j-a_{N-1}^j\right)-\frac{\overline{\lambda}_{N-2}^{j}}{h^2}\left(a_{N-1}^j-a_{N-2}^j\right)+\frac{\partial J}{\partial T_{N-1}^j},$$

$$-\frac{\rho}{\Delta\tau}\left(\overline{c}_N^{j+1}a_N^{j+1}-\overset{+}{c}_N^{j}a_N^j\right)=\frac{\overline{\lambda}_{N-1}^{j}}{h^2}\left(2a_N^j-a_{N-1}^j\right)+\frac{\partial J}{\partial T_N^j},\tag{7}$$

$$a_i^{M+1}=0,\quad i=\overline{3,\ N-2},\quad j=\overline{0,\ M},$$

where

$$\overset{+}{\lambda}_i^j=\lambda_i^j+0.5\left(T_i^j-T_{i+1}^j\right)\left(\lambda_i^j\right)';\quad \overline{\lambda}_i^j=\lambda_i^j-0.5\left(T_i^j-T_{i+1}^j\right)\left(\lambda_i^j\right)';$$

$$\overset{+}{c}_i^j=c_i^j+0.5\left(T_i^j-T_i^{j-1}\right)\left(c_i^j\right)';\quad \overline{c}_i^j=c_i^j-0.5\left(T_i^j-T_i^{j-1}\right)\left(c_i^j\right)';$$

and $(\lambda_i^j)'$ and $(c_i^j)'$ are the derivatives with respect to temperature at the points $0.5(T_i^j + T_{i+1}^j)$ and $0.5(T_i^j + T_i^{j-1})$, respectively.

The system of equations (7) is linear in the conjugate variables a_i^j and is solved by iteration. The gradient functional with respect to the specific heat flux is calculated according to [10]

$$J'(q)=\frac{2}{h}a_i^j,\quad j=\overline{0,\ M}.\tag{8}$$

There also exist other methods of determining the gradient functional [2, 12].

The calculation is performed sequentially. First the direct problem (4) and conjugate problem (7) are solved. Then the resulting values of the conjugate variables are used to calculate the gradient functional from (8), the values β_k and $\mathbf{p}_k$ are determined, then using (6) a new approximation $\mathbf{q}_{k+1}$ is found. In this way the accuracy of the solution of the direct problem is made consistent with the accuracy inherent in the chosen stepsize.

The sequence of values $\{J_k\}$, resulting from the solution of the inverse problem by the iteration procedure described above, satisfies the relaxation relation

$$J(q_{k+1})\leqslant J(q_k).\tag{9}$$

We use the following cutoff criterion

$$\sqrt{J_k}-\sqrt{J_{k+1}}\leqslant\varepsilon.\tag{10}$$

However, condition (10) can only apply when the stepsize in time is chosen such that the corresponding finite-dimensional form of the inverse problem is well determined [2]. Obviously this approximation to the computational model may not always be possible because the accuracy of the final solution may be lowered and it is often necessary to use smaller values of the stepsize $\Delta\tau$. In this case, the nonregular stage of the iterative process usually develops for a number of iterations corresponding to condition (10).

We consider two other methods of cutting off the iteration process and refer to them as cutoffs based on additional measurements and increase of the functional, respectively.

Cutoff with Respect to Additional Measurement. For a given heat flux density $q(\tau)$, at the points $0 \leqslant x_1 \leqslant b$ and $0 \leqslant x_2 \leqslant b$ let there correspond exact (but unknown) temperatures $T_1(\tau)$ and $T_2(\tau)$, respectively. Instead of these temperatures, the real experimental values will always be approximations:

$$\tilde{T}_1(\tau)=T_1(\tau)+\delta_1(\tau),\quad \tilde{T}_2(\tau)=T_2(\tau)+\delta_2(\tau).$$

The disturbances $\delta_1(\tau)$ and $\delta_2(\tau)$ are usually due to various random fluctuation processes. As a rule, $T_1(\tau)$ and $T_2(\tau)$ change in time much more slowly than the low-frequency components of the disturbances.

We assume that the iteration solution of the inverse heat-conduction problem is set up using the temperature $\tilde{T}_1(\tau)$. As shown by the numerical experiments, an increase in the number of iterations initially leads to the successively refined recovery of the slowly varying structural features of the required function $q(\tau)$ corresponding to the curve $T_1(\tau)$. In this case $T(q_k(\tau), x_1, \tau)$ calculated from the solution of the direct problem (1), (2) approximates $T_1(\tau)$. Then there gradually appears and develops additional higher-frequency vibrational components, due to the disturbance $\delta_1(\tau)$ in the function $\tilde{T}_1(\tau)$.

Beginning with a certain iteration, the curves $q_k(\tau)$ deviate more and more from the required solution, adjusted for the perturbed value of the temperature. Therefore, the approximation error $\Delta_k^{T(x_2)} = \left\{\int_0^{\tau_m} [T(q_k(\tau),\ x_2,\ \tau) - T_2(\tau)]^2 d\tau\right\}^{0.5}$, calculated at point x_2 for the exact temperature $T_2(\tau)$, at first decreases with increasing k in the iteration process, but then goes through a minimum and starts to increase. Obviously the value $q_{k^*}(\tau)$ corresponding to this minimum will be the best approximation to $q(\tau)$.

We assume that the random functions $\delta_1(\tau)$ and $\delta_2(\tau)$ are uncorrelated and introduce the error

$$\Delta_k(x_2) = \left\{\int_0^{\tau_m} [T(q_k(\tau),\ x_2,\ \tau) - \tilde{T}_2(\tau)]^2 d\tau\right\}^{0.5} \tag{11}$$

for the perturbed temperatures $\tilde{T}_2(\tau)$ known from experiment. One expects that the minimum of this quantity will be at an iteration number near k^*. Therefore, the first cutoff method will be based on the determination of an iteration number $\tilde{k}$ for which $\Delta_k(x_2)$ is a minimum:

$$\tilde{k} : \min_k \Delta_k(x_2). \tag{12}$$

This method of iteration cutoff for inverse heat-conduction problems was discussed in [13].

Cutoff Condition with Respect to Increment of the Functional. As follows from the above discussion, in the solution of an incorrectly posed problem by iteration, fluctuations appear in the recovered function near the minimum of the functional J. These fluctuations come from the errors in the initial data $T(x_1,\ \tau)$. In order to determine the iteration number where these fluctuations appear, we studied the behavior of the functional J, the increment of the functional $\Delta J_{k+1} = J_{k+1} - J_k$, and also the quantity

$$\Delta_p = \sqrt{(\mathbf{J}_k',\ \Delta\mathbf{q}_k)} - \sqrt{\Delta J_{k+1}}, \tag{13}$$

which represents the distance from the tangent at a point corresponding to the value $\mathbf{q}_k$ on the hyperplane to the surface of the functional at the point $\mathbf{q}_{k+1} = \mathbf{q}_k + \Delta\mathbf{q}_k$. For a convex function $\Delta_p \geqslant 0$, and the equality only occurs at the extremum of the functional.

We considered the recovery of various functions $q(\tau)$ for the linear and nonlinear forms of (1), (2) using data perturbed with different error levels δ. The results of the calculations are discussed below.

The dependence $\sqrt{J}(k)$ is of the relaxation type. The nature of this curve is affected by the location x_1 of the temperature-sensitive element. For values of x_1 far from the heated surface ($\Delta Fo < 0.1$), the curve $\sqrt{J}(k)$ has a sharp shelflike forms starting with an iteration number k^* in a region near the value of the error δ of the initial data. As the value of x_1 gets closer to the heated surface, a gradual transition from a shelflike dependence to a monotonic decrease occurs. We were not able to show the appearance of fluctuations in the solution with the help of the dependence $\sqrt{J}(k)$. Also a study of the behavior of the function $\sqrt{\Delta J}(k)$, which has a saw-toothed form, yielded nothing in this regard.

The calculations showed that when the error is small [$\delta \leqslant 1\%\ T_{max}(x_1)$], a completely satisfactory $q(\tau)$ is obtained using the cutoff condition (10).

Encouraging results were obtained by comparing the behavior of $\sqrt{J}(k)$ and $\Delta_p(k)$. With increase in the number of iterations, the function $\Delta_p(k)$ changes such that at first it decreases to a certain value Δ_p^* and then begins to oscillate, usually with small amplitude. It is noted that when we use temperature data corresponding to a point x_1 for which the discretization step $\Delta Fo \leqslant 0.1$, the iteration process can be stopped if the first minimum of $\Delta_p(k)$ is reached for $\sqrt{J}(k)$ on the shelf. In this case the result has sufficient accuracy.

If we use data at a point near the heated surface, ($\Delta Fo > 0.1$), where $\sqrt{J}(k)$ does not have a sharp shelf, then as shown from the numerical results, in many cases it is possible during the planning stages of the experiment, to establish threshold value Δ_p^*, which can serve as a cutoff condition for the iteration process in the solution of an actual incorrectly posed problem. In particular, for the mathematical model considered here, the threshold value $\Delta_p^* = 0.5$ deg was taken. Usually if the minimization of the functional is continued further, worse results are obtained. The calculations are also showed that in this case a fairly good recovery of $q(\tau)$ is obtained using the cutoff condition (12).

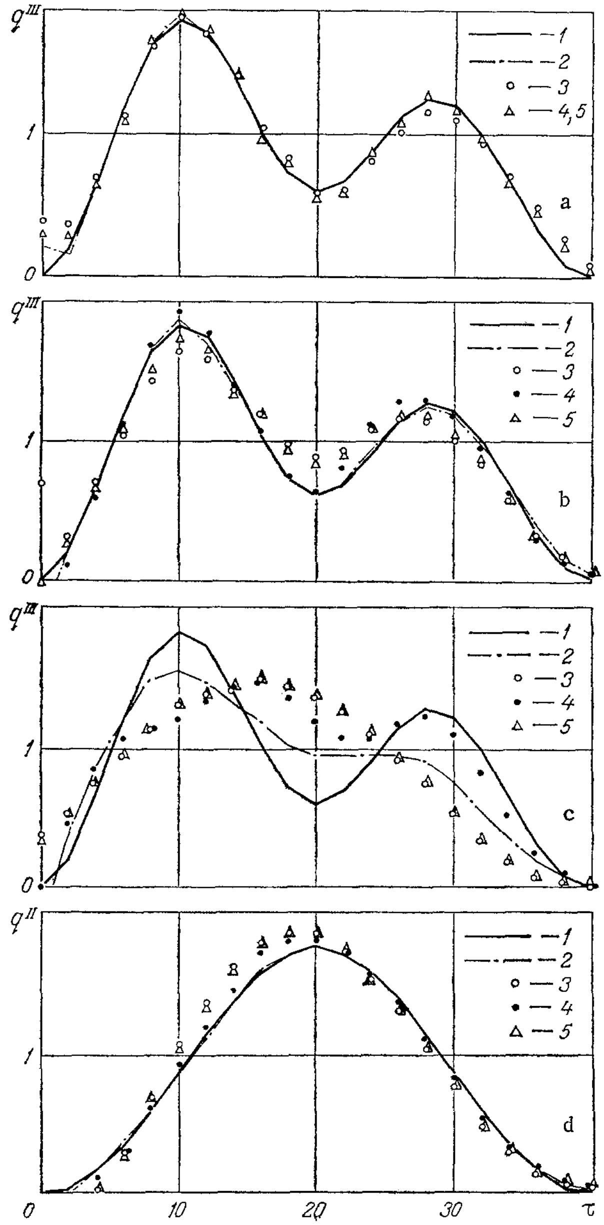

Fig. 1. Recovery of the heat flux density q^{III} (10^6 W/m²) with respect to a temperature-sensitive element at position: a) x_1 = 0.0048 m; b) x_1 = 0.0168 m; c) x_1 = 0.040 m; d) q^{II} (10^6 W/m²) with respect to data from a temperature-sensitive element at x_1 = 0.040 m with the corresponding number of iterations k of the calculation indicated: 1) exact value of the recovered heat flux; 2) recovery for an exactly specified temperature, k = 22; 3) recovery using perturbed data and the cutoff condition $\sqrt{J_k} \approx \delta_{x_1}$, (a) and (b) k = 6; (c) and (d) k = 4; 4) the same, but for the condition $\tilde{k}$: $\min \Delta_k$ (x_2 = 0.0032 m); (a) k = 8; (b) k = 10; (c) k = 19; (d) k = 6; 5) the same, but for the condition $\Delta_p \leqslant 0.5$, (a) k = 8; (b) k = 7; (c) and (d) k = 5.

For illustration of the results considered here of applying the conditions of selecting an iterative solution, we present data from a series of calculations, where various forms of q(τ) were recovered.

In the modeling of the thermal processes, boundary conditions of the second kind were specified in the form of the functions:

$$q^{I}(\tau) = z_1\left(\frac{j-M}{M-1}\right)^2, \tag{14}$$

$$q^{II}(\tau) = z_2\left[1 - \cos\frac{2\pi}{(M-1)}(j-1)\right], \tag{15}$$

$$q^{III}(\tau) = z_2\left(\frac{B-j}{B-1}\right)^2\left\{1 - \cos\frac{4\pi}{(M-1)}(j-1) + 0.5\left[1 - \cos\frac{2\pi}{(M-1)}(j-1)\right]\right\}. \tag{16}$$

From the solution of the direct problem we determine the temperature dependence f(τ) = $T(x_l, \tau)$ at a point with a given coordinate x_l. We modeled the reading of the measuring apparatus by superimposing a perturbation on this dependence imitating the actual decoding and sensitivity errors of the temperature-sensitive elements. Thus, the initial dependence for the inverse heat-conduction problem was calculated according to the equation

$$\hat{T}_l(\tau) = T(x_l, \tau) + \frac{\delta_0\omega_{1l}}{3} + \frac{\delta_1\omega_{2l}^{j}}{3},$$

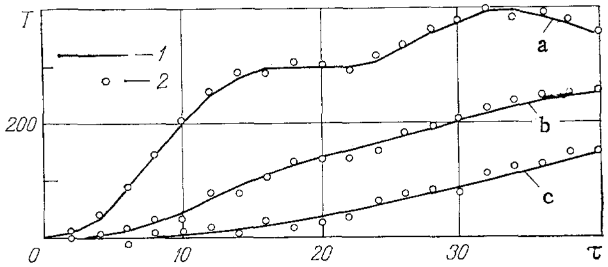

Fig. 2. Temperature dependence $T(x_1, \tau)$ (°C) used for the recovery of the heat flux density q^{III} according to data from temperature-sensitive elements at the points: a) $x_1 = 0.0048$ m; b) 0.0168; c) 0.040 m. Curve 1 is the exact dependence curve 2 is the perturbed temperature dependence.

where $\delta_0 = 0.045 \max(T\dot{J}(x_l))$ is the sensitivity error of the temperature-sensitive element (a constant), and $\delta_1 = 5;\ 12.5;\ 20°$ is the oscillogram decoding error (a variable) ω_{1l} and ω_{2l} are random quantities, distributed normally with a dispersion D = 1 and an expected value M = 0.

The following quantities were specified in the calculations:

In the linear problem $z_1 = 5 \cdot 10^5$ W/m²; $z_2 = 2.5 \cdot 10^5$ W/m²; b = 0.003 m; $\tau_m = 20$ sec, $a = 0.1843 \cdot 10^{-6}$ m²/sec.

The coordinates of the temperature-sensitive elements were chosen such that the discretization step $\Delta\tau = 0.5$ sec would satisfy the condition [2] $0.01 < \Delta Fo < 0.1$ for all points $0 < x_l \leqslant b$.

As initial data, we took the temperature dependence at the points $x_1 = 0.003$ m; 0.00144 m; 0.00096 m, corresponding to the dimensionless discretization steps ΔFo = 0.0102, 0.0444, 0.1.

For the nonlinear problem we used $z_1 = 2.91 \cdot 10^6$ W/m²; $z_2 = 0.833 \cdot 10^6$ W/m²; b = 0.040 m; $\tau_m = 40$ sec; $\Delta\tau = 2$ sec; $x_1 = 0.040$ m; 0.0168 m; 0.0048 m.

We considered an St25 plate with thermal diffusivity $0.886 \cdot 10^{-5} \leqslant a \leqslant 1.416 \cdot 10^{-5}$ m²/sec. The discretization steps in time and position of the temperature-sensitive element were chosen from the condition $0.01 < \Delta Fo < 1.3$. Then the discretization steps corresponding to the positions used were $0.0111 \leqslant \Delta Fo \leqslant 0.0177$; $0.063 \leqslant \Delta Fo \leqslant 0.1$; $0.762 \leqslant \Delta Fo \leqslant 1.229$.

The number of grid points in the coordinate x was chosen from the quantity ΔFo, according to [14].

The error criterion was calculated according to (11) in the method based on additional measurements. The solution was selected according to condition (12) with the use of data from a temperature-sensitive element at the following positions: in the linear case $x_2 = 2.76 \cdot 10^{-3}$ m; $1.80 \cdot 10^{-3}$ m; $0.84 \cdot 10^{-3}$ m; in the nonlinear case $x_2 = 40 \cdot 10^{-3}$ m; $36 \cdot 10^{-3}$ m; $3.2 \cdot 10^{-3}$ m.

As an illustration of the results, we show in Fig. 1 the recovery of the dependence (16) in the nonlinear case using data from a temperature-sensitive element at various positions x . For comparison we show the curve corresponding to the number of iterations in which the error cutoff condition

$$\sqrt{J_k} \approx \delta_{x_1} \qquad (17)$$

for the temperature-sensitive element is satisfied. In Fig. 2 we show the dependence $T(x_1, \tau)$ used for the recovery of the function (16). It should be noted that when the position of the measuring temperature-sensitive element is far enough away from the heated surface, the accuracy in the recovery of a complicated curve can be impaired significantly. In particular, such a situation is illustrated in Fig. 1c for $x_1 = 0.040$ m. For a simpler dependence, the required function can be recovered much more accurately with other cutoff conditions, as shown in Fig. 1d.

Hence, the numerical experiments indicate that the above-discussed empirical methods of choosing approximate solutions of incorrectly posed inverse problems can be applied successfully in practice.

NOTATION

T, temperature; γ, specific density of the material; $\lambda(T)$, thermal conductivity; c(T), specific heat; a, thermal diffusivity; τ, time; x, spatial coordinate; $\Delta\tau$, grid stepsize in time; h, grid stepsize in position; ΔFo, dimensionless computational discretization step of the function $q(\tau)$; b, thickness of the plate; τ_m, time of the experiment; l, number of the additional temperature-sensitive element; δ, error in specifying the initial information; x_l, distance between temperature-sensitive element l and the heated surface; k, iteration number for minimization of the functional; s, iteration number for parabolic minimization; j, M, current time index and number of time grid points; i, N, current coordinate index and number of grid points with respect to coordinate; β, slope stepsize; p, slope direction.

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